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The Asymptotic Connectivity of Labelled Regular Graphs

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It is shown that almost all labelled r-regular graphs are r-connected, for any fixed $r \geqslant 3$. This is an instance of a more general result which applies to labelled locally restricted graphs whose points have degrees lying between r and R, where r and R are fixed, $3 \leqslant r \leqslant R$. The proof employs a known asymptotic formula for the numbers of these locally restricted graphs. It is also demonstrated that for fixed $k > 0$ and $r \geqslant 3$, almost all labelled r-regular graphs with girth at least $\lceil k/(r-2) \rceil$ are cyclically k-connected. This provides an asymptotic formula for the number of labelled cyclically k-connected r-regular graphs with p points, for fixed k and r.

1. INTRODUCTION

A *k-point-cutset* of a graph G is a set of k points whose removal from G leaves a disconnected graph. If G has at least $k+1$ points and has no $(k-1)$-point-cutset, then G is *k-connected.* Also, G is *r-regular* if all its points have degree r. A 3-regular graph is *cubic.* Our main objective is to show that for any fixed $r \geqslant 3$, almost all labelled r-regular graphs are r-connected. This is not true for $r = 1$ or 2. Here, as in the rest of this paper, the statement that almost all labelled graphs with property X have property Y is interpreted as follows. Let $X(p)$ denote the number of labelled graphs on p points with property X, and $Y(p)$ the number of these which also have property Y. Then $Y(p)/X(p)$ approaches 1 as p approaches infinity, where p takes only those values for which $X(p)$ is non-zero.

The *partition* of a graph is the collection of degrees of its points. *Locally restricted* graphs are graphs with a given partition. In a labelled locally restricted graph, the degree of the point labelled i is not specified for any

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given i, but the total number of points with degree d is specified for all d. In Section 2, we consider locally restricted graphs whose degrees lie between r and R, where $3 \leqslant r \leqslant R$. Theorem 1 is essentially that almost all such graphs are r-connected. In other words, if for each possible partition D of a graph with degrees lying between r and R we define the probability $\tau(D)$ that a graph with partition D is r-connected, then $\tau(D)$ approaches 1 as p approaches infinity (irrespective of precisely which partitions D are considered). By putting $r = R$, we obtain our main objective, Corollary 1.1. By contrast, for $r = 1$ and 2 it is observed that almost all labelled r-regular graphs are disconnected.

A graph G has *connectivity* k if it is k-connected but not $(k+1)$-connected. It is immediate that all points of a k-connected graph have degrees at least k. Corollary 1.2 is essentially that for almost all labelled graphs G with degrees in the range $3,..., R$, the connectivity of G equals the minimum degree of a point in G. This resembles the well-known result of Erdös and Rényi [3] that if $q(p) = \frac{1}{2}p \log p + \alpha p + o(p)$, α a real constant, then almost all labelled graphs with p points and $q(p)$ lines have this same property, of connectivity being equal to minimum degree.

A set S of lines (or points) of a graph G is *cycle-separating* if the removal from G of all lines (points) in S leaves a disconnected graph, at least two components of which contain cycles. G is *cyclically k-connected* (*cyclically k-point-connected*) if it has no cycle-separating set of j lines (points) for any $j < k$. In Section 3 we consider the numbers of labelled r-regular graphs which are cyclically k-connected or cyclically k-point-connected, and find that for $r \geqslant 3$ these are asymptotically equal to the numbers of labelled r-regular graphs with girth at least $\lceil k/(r-2) \rceil$, where $\lceil \ \rceil$ denotes the ceiling function.

The proofs in the present paper rely on the following result of Bender and Canfield [1]. Let $\rho(r, p)$ denote the number of labelled r-regular graphs on p points. Then as p approaches infinity with p even if r is odd,

$$\rho(r, p) \sim \frac{(rp)! \exp((1-r^2)/4)}{(rp/2)!\, 2^{rp/2}(r!)^p}, \tag{1.1}$$

where $r \geqslant 1$ is fixed. This is an instance of the following more general formula, for graphs with given partition, which is implied directly from the results in [1]. Suppose the non-negative integers p, q, R, and $g(0),..., g(R)$ satisfy

$$p = \sum_{i=0}^{R} g(i) \qquad \text{and} \qquad 2q = \sum_{i=1}^{R} ig(i).$$

Then whatever the values of p and the $g(i)$ subject to these conditions, the number of labelled graphs with p points, q lines, and precisely $g(i)$ points of degree i for $0 \leqslant i \leqslant R$ is asymptotic to

$$\frac{(2q)!\exp(-\alpha-\alpha^2)\,p!}{q!\,2^q\prod_{i=0}^{R}(g(i)!(i!)^{g(i)})} \tag{1.2}$$

as q approaches infinity with R fixed, where

$$\alpha=(4q)^{-1}\sum_{i=2}^{R} i(i-1)\,g(i).$$

Note that α is bounded above by $(R-1)/2$.

The results of [7] relate to the asymptotic numbers of labelled regular graphs with given numbers of cycles of various lengths. In particular, for $j \geqslant 4$, the number of labelled r-regular graphs of girth at least j with p points, where p is even if r is odd, is asymptotic to

$$\rho(r,p)\exp\left(-\sum_{i=3}^{j-1}\frac{(r-1)^i}{2i}\right) \tag{1.3}$$

as p approaches infinity with r and j fixed.

We shall use $B(n, r)$ to denote the binomial coefficient $n!((n-r)!\,r!)$. Graph theoretic notation not defined here can be found in the book by Harary [4].

2. Locally Restricted Graphs

Our first aim is to prove a lemma which is used in establishing both Theorem 1 in this section and Theorem 2 in the next section. It is worthwhile to begin by contemplating the well-known fact that for all positive k, almost all labelled graphs are k-connected. One of the many ways to prove this is to argue along the following lines. Suppose G is a labelled graph on p points in which the points labelled $1,\dots, k$ form a k-point-cutset. If we slice each of these k points in two, G is split into two graphs F and H with t and $p-t+k$ points, respectively. The number of possibilities for the graphs F and H, when properly labelled, is at most $2^{B(t,2)+B(p-t+k,2)}$. But of the labels $k+1,\dots, p$, some are used in F and the rest in H, so we must multiply by $B(p-k, t-k)$ to obtain an upper bound on the number of possibilities for G. It follows that the number of labelled graphs on p points which have some k-point-cutset is at most

$$\sum B(p,k)\,B(p-k,t-k)\,2^{B(t,2)+B(p-t+k,2)},$$

where the sum is for $k+1 \leqslant t \leqslant p-k-1$. It can be shown that as p approaches infinity, this sum is negligible compared with the total number $2^{B(p,2)}$ of labelled graphs on p points. Hence almost all labelled graphs are $(k+1)$-connected. Our proof of the forthcoming Theorem 1 can be viewed as an adaptation of this argument, to locally restricted graphs.

Unless otherwise specified, all summations and products are taken for $i=0,\dots,R$. Suppose $3 \leqslant r \leqslant R$, and let $D=(g(r), g(r+1),\dots, g(R))$ and $g(i)=0$ for $i<r$, with

$$p=\sum g(i). \tag{2.1}$$

Let $L(D)$ denote the set of labelled graphs on p points with $g(i)$ points of degree i for $r \leqslant i \leqslant R$, and hence no points with other degrees, and put $\nu(D)=|L(D)|$. Suppose G is a graph in $L(D)$ in which the points $1, 2,\dots, k$ form a k-point-cutset S for some $k \geqslant 2$. Let F' be a component of $G-S$ such that the sum of the number of lines in F' and the number of lines joining F' to S is minimised, and define F to be the graph formed from G by retaining only those points in F' and S, and only those lines in F' or joining F' to S. In the following lemma, the notation $O(\ )$ denotes a bound depending only on R and k, and not on r, p, m, n or the particular choice of D satisfying (2.1).

Let $\nu(D, m, n)$ denote the number of possibilities for G in which F can have m points and n lines.

LEMMA. *The number $\nu(D, m, n)$ is*

$$p^{m-n-k}\nu(D)\,O(1)$$

for $n<24(R+k)$, and

$$p^{2-r-2k}\nu(D)\,O(1)$$

for $n \geqslant 24(R+k)$.

Proof. It can be assumed that $L(D)$ is nonempty, so that

$$2q=\sum ig(i),$$

where G has q lines. It follows from the minimality of n that

$$n \leqslant q/2. \tag{2.2}$$

Let H be the graph formed by removing all points in F' from G. Each point or line of G is contained in either F or H, the only duplications being the points in S. As the points of G in S are labelled $1,\dots, k$, the labelling of G

induces a labelling of the graph $F \cup H$ using the labels $1,\dots,p$ such that the points of S are labelled $1,\dots,k$ in both F and H, no other labels being duplicated. We denote this improperly labelled version of $F \cup H$ by G'. The number $\nu(D, m, n)$ is just the number of possibilities for G'.

For $0 \leqslant i \leqslant R$, suppose $s(i)$ $(s'(i), s''(i))$ is the number of points in the set S which have degree i in $G(F, H)$. Also, suppose $f(i)$ $(h(i))$ is the number of points of $F(H)$ which have degree i. Let μ be the number of possibilities for G' under these conditions, so that μ depends on D, m, n and the $s(i)$, $s'(i)$, $s''(i)$, $f(i)$ and $h(i)$. By (1.2), the number of labelled possibilities for F under these conditions is

$$\frac{(2n)!\, m!\, O(1)}{n!\, 2^n \prod((i!)^{f(i)} f(i)!)}.$$

Moreover, there are potentially

$$\prod B(f(i), s'(i))$$

ways to choose the points of F which constitute the set S. Specifying that these points have the labels $1,\dots,k$ divides the number of possibilities by $B(m, k)$. Hence, the number of labelled possibilities for F in which the points of S are labelled $1,\dots,k$ is

$$\frac{(2n)!(m-k)!\, O(1)}{n!\, 2^n \prod((i!)^{f(i)}(f(i)-s'(i))!)}. \tag{2.3}$$

Similarly, the number of labelled possibilities for H in which the points of S are labelled $1,\dots,k$ is

$$\frac{(2q-2n)!(p-m)!\, O(1)}{(q-n)!\, 2^{q-n} \prod((i!)^{h(i)}(h(i)-s''(i))!)}. \tag{2.4}$$

An expression for the number of possibilities for the graph G' is obtained by multiplying the product of (2.3) and (2.4) by $B(p-k, m-k)$. Hence,

$$\mu = \frac{(2n)!(2q-2n)!(m-k)!(p-m)!\, B(p-k, m-k)\, O(1)}{n!(q-n)!\, 2^q \prod((i!)^{f(i)+h(i)}(f(i)-s'(i))!(h(i)-s''(i))!)}.$$

When (1.2) is used for an asymptotic estimate of $\nu(D)$, this becomes

$$\mu = \frac{O(1)\, \nu(D)\, B(q, n)}{p^k B(2q, 2n)} \prod \frac{g(i)!}{(f(i)-s'(i))!(h(i)-s''(i))!}. \tag{2.5}$$

As

$$f(i) - s'(i) + h(i) - s''(i) = g(i) - s(i), \tag{2.6}$$

and $\sum s(i) = k$, it follows that

$$\prod \frac{g(i)!}{(f(i) - s'(i))!(h(i) - s''(i))!} \leqslant p^k \prod B(g(i), f(i) - s'(i))$$
$$\leqslant p^k B(p, m-k),$$

where truth of the latter inequality also depends on $\sum g(i) = p$ and

$$\sum (f(i) - s'(i)) = m - k.$$

Since $r \geqslant 3$ we have $q > p$, so that (2.5) now becomes

$$\mu = \nu(D)\, B(q, n)\, B(2q, 2n)^{-1}\, B(q, m-k)\, O(1). \tag{2.7}$$

To obtain the total number of possibilities for the graph G', one must sum the number μ over all possibilities for the numbers $s(i)$, $s'(i)$, $s''(i)$, $f(i)$, $h(i)$, for $0 \leqslant i \leqslant R$. As F has m points, the number of possibilities for $f(0), \ldots, f(R)$ is at most m^{R+1}. There are $O(1)$ possibilities for the $s(i)$, $s'(i)$ and $s''(i)$. When all these numbers have been chosen, the $h(i)$ are determined by (2.6). Hence, by (2.7) together with the fact that $\nu(D, m, n)$ is the number of possibilities for G', we have

$$\nu(D, m, n) = m^{R+1} B(q, n)\, B(2q, 2n)^{-1}\, B(q, m-k)\, \nu(D)\, O(1). \tag{2.8}$$

Consider $n < 24(R + k)$. Then m and n are both $O(1)$ and so (2.8) implies

$$\nu(D, m, n) = q^{m-n-1} \nu(D)\, O(1). \tag{2.9}$$

Now suppose $n \geqslant 24(R + k)$. The points of F which are not in S have degree at least r, so

$$m - k \leqslant 2n/r \leqslant 2n/3. \tag{2.10}$$

In view of (2.2), therefore, $m - k$ can be replaced in (2.8) by $\lfloor 2n/3 \rfloor$. An application of Stirling's formula to (2.8) then yields

$$\nu(D, m, n) = m^{R+1} q^{1/2} n^{-1/2} (q-n)^{-1/2}\, B(q, n)^{-1}$$
$$\times B(q, \lfloor 2n/3 \rfloor)\, \nu(D)\, O(1). \tag{2.11}$$

Note that $m \leqslant q$, so that

$$m^{R+1} q^{1/2} n^{-1/2} (q-n)^{-1/2} \leqslant q^{R+2}.$$

Also it is readily verified that

$$B(q, n)^{-1} B(q, \lfloor 2n/3 \rfloor) \leqslant (n/(q - \lfloor 2n/3 \rfloor))^{n/3}.$$

Hence, (2.11) implies

$$\nu(D, m, n) = q^{R+2}(n/(q - \lfloor 2n/3 \rfloor))^{n/3}\, \nu(D)\, O(1). \tag{2.12}$$

For $n < \sqrt{q}$,

$$(n/(q - \lfloor 2n/3 \rfloor))^{n/3} \leqslant (4/q)^{n/6} \leqslant q^{-n/12}$$

when $q > 16$, and so (2.12) implies

$$\nu(D, m, n) = q^{2-r-2k} \nu(D)\, O(1), \tag{2.13}$$

as $n \geqslant 24(R + k)$ and $r \leqslant R$. On the other hand, for $n \geqslant \sqrt{q}$,

$$(n/(q - \lfloor 2n/3 \rfloor))^{n/3} \leqslant (3/4)^{\sqrt{q}/3}$$

and so (2.12) implies (2.13) in this case also. The Lemma follows by (2.9) and (2.13) combined as $q \geqslant p$ and $2 - r - 2k$ is negative.

We are now ready to prove the main result of this section.

THEOREM 1. *The number of r-connected graphs in $L(D)$ is $\nu(D)(1 - O(p^{2-r}))$, where $O(\ )$ denotes a bound depending only on R.*

Proof. Let G be an element of $L(D)$ rooted at an $(r-1)$-point-cutset S. If r is 3 and G has at least one component isomorphic to K_4, we specify that S consists of two points in one such component. The number of possibilities for G is then an upper bound on the number of elements of $L(D)$ which are not r-connected.

Define a subgraph F of G as before. By the Lemma, the number of possibilities for G in which the points of S are labelled $1, \ldots, r-1$ and F has m points and n lines is

$$p^{m-n+1-r} \nu(D)\, O(1)$$

for $n < 24(R + r - 1)$, and

$$p^{4-3r} \nu(D)\, O(1)$$

for $n \geqslant 24(R + r - 1)$. To relax the condition on the labels of the points in S, we must multiply these numbers by $B(p, r-1) = p^{r-1} O(1)$. Hence, as the bound denoted by $O(\ )$ in the Lemma is independent of m and n, the total number of possibilities for G is

$$\nu(D)\, O(1) \left(\sum p^{m-n} + \sum p^{3-2r}\right), \tag{2.14}$$

where the first summation is over all $n < 24(R + r - 1)$ and all relevant values of m (that is, all m for which $\nu(D, m, n)$ is non-zero), and the second summation is over all $n \geqslant 24(R + r - 1)$ and all relevant m. There are clearly $O(p^2)$ possible values of the ordered pair (m, n). Since $r \geqslant 3$, it follows that the sum $\sum p^{3-2r}$ in (2.14) is $p^{2-r}O(1)$. On the other hand, the sum $\sum p^{m-n}$ contains $O(1)$ terms, and it will shortly be shown that $m - n \leqslant 2 - r$ whenever $\nu(D, m, n)$ is non-zero. Hence, (2.14) is

$$p^{2-r}\nu(D)\, O(1),$$

from which the theorem follows.

It only remains to show $m - n \leqslant 2 - r$. Put $m = r - 1 + t$, so that t is the number of points of F not in S. Each such point has degree at least r, and thus $n \geqslant \frac{1}{2}rt$, which implies

$$m - n \leqslant \tfrac{1}{2}(t - 2)(2 - r) + 1.$$

If $t \geqslant 5$ this implies $m - n \leqslant 2 - r$ as required, since m and n are integers and $r \geqslant 3$. Since F has no loops, it is impossible that t is 1, which leaves $t = 2$, 3 and 4. As there are at most $\frac{1}{2}t(t - 1)$ lines between the points of F not in S, it follows that $n \geqslant tr - \frac{1}{2}t(t - 1)$, and hence

$$m - n \leqslant (t - 1)(\tfrac{1}{2}t + 1 - r).$$

For $2 \leqslant t \leqslant 4$, this implies $m - n \leqslant 2 - r$ in all cases except when t is 4 and r is 3, which is only a problem when m and n are both 6. In this case, the graph F must consist of a copy of K_4 together with the two isolated points in S. However, this is impossible because of the special condition originally imposed on S when G has a component isomorphic to K_4. The proof is complete.

Theorem 1 implies the result foreshadowed in [6], that almost all labelled cubic graphs are 3-connected. In fact, this can be generalised as follows.

COROLLARY 1.1. *For fixed $r \geqslant 3$, almost all labelled r-regular graphs are r-connected.*

Equation (1.1) therefore provides a valid asymptotic formula for the number of labelled r-connected r-regular graphs on p points, for $r \geqslant 3$.

The situation is different for $r = 1$ and 2, when almost all labelled r-regular graphs are in fact disconnected. This is immediate for 1-regular graphs, since all such graphs with more than two points are disconnected. On the other hand, the only connected 2-regular graph on p points is the p-cycle, which is 2-connected and can be labelled in $p!/(2p)$ ways. By (1.1) and Stirling's formula, it follows that the proportion of labelled 2-regular graphs on p points which are connected (or 2-connected) is asymptotic to $\frac{1}{2}e^{3/4}(\pi/p)^{1/2}$.

Since $O(\ )$ in Theorem 1 is dependent only upon R, we can sum over any number of sequences D. In the following example, all permissible sequences D are taken into account.

COROLLARY 1.2. *Let $T(p, R)$ be the set of labelled graphs on p points with degrees in the range* 3,..., R, *and let $\tau(p, R)$ be the proportion of graphs G in $T(p, R)$ for which the connectivity of G is equal to the minimum degree of its points.*

Then

$$\tau(p, R) \sim 1$$

as p approaches infinity with R fixed.

The following problem is suggested by Theorem 1.

PROBLEM. What conditions on a function $f(p)$ are necessary and sufficient for $R = f(p)$ to imply that the proportion of graphs in $L(D)$ which are r-connected approaches 1 as $p \to \infty$?

From Theorem 1, it is sufficient if $f(p)$ is bounded above by a constant, but this is almost certainly not necessary. In the other direction, it is easy to show that $f(p) \leqslant p-1$ is not sufficient. Indeed, sometimes the set $L(D)$ consists of the labelled versions of a unique labelled graph, which is moreover not r-connected. This was suggested to the author by P. Erdös as being a strong possibility. An example is provided, for all even $p \geqslant 6$, by the graph with two points of degree $p-1$ and the rest of degree 3. The points of large degree comprise a 2-point-cutset. Substantial improvements on these results are beyond the scope of our present methods.

The proof of Theorem 1 can be modified so as to obtain stronger bounds on the numbers of graphs in $L(D)$ which are not k-connected, for $k < r$. In case k is 1 and attention is restricted to r-regular graphs, the theory developed by Wright [8] can be applied to find an asymptotic expansion for these numbers.

3. CYCLICALLY k-CONNECTED r-REGULAR GRAPHS

The Lemma can be applied to the asymptotic enumeration of labelled locally restricted graphs which are cyclically k-connected or cyclically k-point-connected. Results are most readily obtained when the graphs are regular, and we restrict our attention to this case. Indeed, many authors consider the concept of cyclic k-connectedness only in connection with cubic graphs and $k=4$ (for example, see [2, Chap. 7; 5]).

THEOREM 2. *For any fixed $r \geqslant 3$ and $k > 0$, the numbers of labelled r-regular graphs on p points which are*

(i) *cyclically k-point-connected,*

(ii) *cyclically k-connected,*

(iii) *of girth at least $\lceil k/(r-2) \rceil$*

are asymptotically equal as p approaches infinity, where p is even if r is odd.

Proof. Firstly, it can be seen that all cyclically k-point-connected r-regular graphs with a sufficiently large number of points are cyclically k-connected.

Secondly, if an r-regular graph G has a cycle C of length $i < \lceil k/(r-2) \rceil$, the set S of lines joining points in C to points not in C has cardinality less than k. The removal of all lines in S from G produces a graph H, one of whose components is C. The rest of H must contain a cycle whenever H has at least $2i$ points. Hence, when p is sufficiently large, every cyclically k-connected r-regular graph on p points has girth at least $\lceil k/(r-2) \rceil$.

The theorem is established by showing, thirdly, that almost all labelled r-regular graphs with girth at least $\lceil k/(r-2) \rceil$ are cyclically k-point-connected. For our purposes here, the notation $O(\ )$ denotes a bound depending only on r and k.

Suppose G is an r-regular graph on p points with girth at least $\lceil k/(r-2) \rceil$ which is cyclically j-point-connected but not cyclically $(j+1)$-point-connected, for some $0 \leqslant j < k$. Clearly G cannot be $(j+1)$-connected. Thus, if $j < r$, Theorem 1 implies that the number of labelled possibilities for G is $\rho(r,p)\,p^{-1}O(1)$, as $\nu(D)$ is $\rho(r,p)$ in this case. On the other hand, if $j \geqslant r$, form the graph G^* by rooting G at a cycle-separating j-point-cutset S. Let F' be a cycle-containing component of $G^* - S$ for which the total number of lines in F' or joining F' with S is minimised, and is hence at most $\frac{1}{2}rp$. Define F, m and n as for the Lemma in Section 2. The conditions of the Lemma are now satisfied, with $g(r)=p$ and $g(i)=0$ for all other i. It will shortly be shown that m is always less than n in the present situation. By an application of the Lemma as in the proof of Theorem 1, it follows that the total number of labelled possibilities for G^*, for all possible m and n with a given p, is $\nu(D)\,p^{-1}O(1) = \rho(r,p)\,p^{-1}O(1)$.

The number of labelled possibilities for G is hence $\rho(r,p)\,p^{-1}O(1)$ for each possible j. As k is fixed, we can sum for $0 \leqslant j < k$, so that the number of labelled r-regular graphs with girth at least $\lceil k/(r-2) \rceil$ which are *not* cyclically k-point-connected is $\rho(r,p)\,p^{-1}O(1)$. In view of the asymptotic formula (1.3), this implies that almost all labelled r-regular graphs with girth at least $\lceil k/(r-2)\rceil$ are cyclically k-point-connected, as required.

It only remains to show $m < n$. Since G is cyclically j-point-connected and S is cycle-separating, each point of S must be adjacent to some point of the graph F', which contains some cycle C. Hence, F is connected and contains the cycle C. Consequently $m \leqslant n$, and $m = n$ only if F is unicyclic. In the latter case, each of the j points in S must be an endpoint in F, and thus the number of lines in F is $\frac{1}{2}(tr + j)$, where t denotes the number of points in F'. We now have

$$j + t = m = n = \tfrac{1}{2}(tr + j)$$

and therefore

$$t = j/(r-2) < \lceil (j+1)/(r-2)\rceil.$$

But the cycle C can have length at most t, contradicting the hypothesised girth of G. Thus $m < n$ and the proof is complete.

Theorem 2 provides an asymptotic formula for the number of labelled cyclically k-connected (or k-point-connected) r-regular graphs via (1.3). The first interesting application is the following.

COROLLARY 2.1. *The number of labelled cyclically* 4-*connected cubic graphs on* $2n$ *points is asymptotic to*

$$(6n)!\, \exp(-10/3)/((3n)!\, 288^n)$$

as n approaches infinity. Hence, the proportion of labelled cubic graphs on $2n$ *points which are cyclically* 4-*connected is asymptotic to* $\exp(-4/3)$ *as n approaches infinity.*

A recurrence relation satisfied by the exact numbers of labelled cyclically 4-connected cubic graphs will be derived elsewhere.

Finally, in view of the results given in [7] regarding the asymptotic numbers of labelled pseudographs, the methods of the present paper can presumably be extended to handle loops and multiple lines.

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